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A GAME OF OPTIMAL PURSUIT OF ONE OBJECT BY SEVERAL†

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A differential game in which m dynamical objects pursue a single one is investigated. All the players perform simple motions. The termination time of the game is fixed. The controls of the first k ($k \leqslant m$) pursuers are subject to integral constraints and the controls of the other pursuers and the evader are subject to geometric constraints. The payoff of the game is the distance between the evader and the closest pursuer at the instant the game is over. Optimal strategies for the players are constructed and the value of the game is found. © 1998 Elsevier Science Ltd. All rights reserved.

1. STATEMENT OF THE PROBLEM

The motions in R^n of pursuers P_i and an evader E are described by the equations

$$P_i: \quad \dot{x}_i = u_i, \quad x_i(0) = x_{i0} \tag{1.1}$$

$$E: \quad \dot{y} = \upsilon, \quad y(0) = y_0 \tag{1.2}$$

where $x_i, u_i, y, \upsilon \in R^n$, u_i is the control parameter of pursuer P_i, and υ is the control parameter of the evader E; throughout, $i = 1, 2, \ldots, m$.

Definition 1. A measurable function $u_j = u_j(t)$, $0 \leqslant t \leqslant \vartheta$ satisfying the constraint

$$\int_0^{\vartheta} |u_j(t)|^2\, dt \leqslant \rho_j^2 \text{ for } j = 1, 2, \ldots, k \tag{1.3}$$

$$|u_j(t)| \leqslant \rho_j \text{ for } j = k+1, \ldots, m \tag{1.4}$$

is called an admissible control of the pursuer P_j, where ϑ is a given fixed instant of time, ρ_j are given positive numbers and k is a non-negative integer.

Definition 2. A measurable function $\upsilon = \upsilon(t)$, $0 \leqslant t \leqslant \vartheta$ satisfying the constraint $|\upsilon(t)| \leqslant \sigma$ is called an admissible control of the evader E. If $u_i = u(t)$ and $\upsilon = \upsilon(t)$, $0 \leqslant t \leqslant \vartheta$ are admissible controls of the pursuer P_i and the evader E, respectively, then the trajectory of the pursuer $x_i(t)$, $0 \leqslant t \leqslant \vartheta$ is defined as an absolutely continuous solution of the Cauchy problem (1.1), and a trajectory of the evader $y(t)$, $0 \leqslant t \leqslant \vartheta$, as an absolutely continuous solution of the Cauchy problem (1.2).

Let $H(x, r)$ ($S(x, r)$) denote a ball (sphere) with centre at x and radius r.

Definition 3. A function $U_j(x, y, \upsilon)$
U_j: $R^n \times R^n \times R^n \to R^n$ for $j = 1, \ldots, k$
U_j: $R^n \times R^n \times H(0, \sigma) \to H(0, \rho_j)$ for $j = k + 1, \ldots, m$
for which the system

$$\dot{x}_j = U_j(x_j, y, \upsilon(t)), \quad x_j(0) = x_{j0}$$
$$\dot{y} = \upsilon(t), \quad y(0) = y_0$$

has a unique absolutely continuous solution for any admissible control $\upsilon(t)$, $0 \leqslant t \leqslant \vartheta$, of the evader E is called a strategy of the pursuer P_j. A strategy U_i is said to be admissible if every control generating it is admissible.

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Definition 4. The strategies U_{i0} of the pursuers P_i, respectively, are said to be optimal if

$$\inf_{U_1,\ldots,U_m} \Gamma_1(U_1,\ldots,U_m)=\Gamma_1(U_{10},\ldots,U_{m0})$$

where

$$\Gamma_1(U_1,\ldots,U_m)=\sup_{\upsilon(\cdot)} \min_{1\leqslant i\leqslant m} |x_i(\vartheta)-y(\vartheta)|$$

U_i are admissible strategies of the pursuers P_i, respectively, and $\upsilon(\cdot)$ is an admissible control of the evader E.

Definition 5. A function $V(x_1,\ldots,x_m,y)$, $V: R^n\times\ldots\times R^n\to H(0,\sigma)$ for which the system

$$\dot{x}_i=u_i(t),\quad x_i(0)=x_{i0}$$
$$\dot{y}=V(x_1,\ldots,x_m,y),\quad y(0)=y_0$$

has a unique absolutely continuous solution for any admissible controls $u_i(t)$, $0\leqslant t\leqslant\vartheta$ of the pursuers P_i is called a strategy of the evader E. If each control generating the strategy V is admissible, the strategy V is said to be admissible.

Definition 6. A strategy V_0 of the evader E is said to be optimal if

$$\sup_V \Gamma_2(V)=\Gamma(V_0)$$

where

$$\Gamma_2(V)=\inf_{u_1(\cdot),\ldots,u_m(\cdot)} \min_{1\leqslant i\leqslant m} |x_i(\vartheta)-y(\vartheta)|$$

and $u_i(\cdot)$ are admissible controls of the pursuers P_i.

If $\Gamma_1(U_{10},\ldots,U_{m0})=\Gamma_2(V_0)=\gamma$, we will say that the game has a value γ [1].

It is required to find the optimal strategies U_{i0} and V_0 of the players P_i and E, respectively, and the value of the game.

Analogous problems have been investigated in many publications. Among the cases that have been considered are, for example; $k=0$, $m=2$ [2]; $k=0$ [3]; $k=0$, $m\leqslant n$ (where n is the dimension of the space) [4].

This paper will develop a method used previously.†

2. THE OPTIMAL APPROACH OF m PURSUERS TO A SINGLE EVADER

Consider the differential game (1.1), (1.2). It can be verified that the reachable domain of the pursuer P_i from the initial position x_{j0} up to time ϑ is the ball

$$H(x_{j0},\rho_j\sqrt{\vartheta})\quad\text{for}\quad j=1,\ldots,k,\quad H(x_{j0},\rho_j\vartheta)\quad\text{for}\quad j=k+1,\ldots$$

Let

$$G_j(l)=H(x_{j0},\rho_j\sqrt{\vartheta}+l)\quad\text{for}\quad j=1,\ldots,k$$
$$G_j(l)=H(x_{j0}\rho_j\vartheta+l)\quad\text{for}\quad j=k+1,\ldots,m$$
$$\gamma=\min\left\{l\geqslant 0:\ H(y_0,\sigma\vartheta)\subset\bigcup_1^m G_i(l)\right\}\tag{2.1}$$

†IBRAGIMOV, G. I., *The optimal approach of two pursuers to a single evader*. Moscow, 1987, 16pp. Deposited at Vsesoyuz. Inst. Nauch. Tekhn. Informatsii, 1987, 5384–B87.

Theorem. If $\sigma\vartheta \leqslant \rho_j\vartheta + \gamma$ ($j = k+1, \ldots, m$) and $(y_0 - x_{i0}, p_0) \geqslant 0$ for some non-zero vector p_0 defined by formula (2.1), then the number γ is the value of the differential game (1.1), (1.2).

The proof of the theorem is based on several lemmas.

Suppose C is the boundary of a closed bounded set $D \subset R^n$, and $X_i = \{x: x \in R^n, (x, p_i) \leqslant d_i\}$ are certain half-spaces, where p_i are given unit vectors and d_i are given numbers.

Lemma 1. Let p_0 be a non-zero vector such that $(p_0, p_i) \leqslant d_i$ and let $C \subset \cup X_i$. Then $D \subset \cup X_i$.

Proof. It will suffice to consider the case when $\operatorname{int} D \neq \varnothing$. Suppose the contrary. Let $y \in \operatorname{int} D$. Hence it follows that $\bar{y}$ lies in none of the half-spaces X_i, i.e. $\bar{y} \notin \cup X_i$. Then no point of the half-line $y(t) = \bar{y} + tp_0$, $t \geqslant 0$ lies in the set $\cup X_i$, because $(y(t), p_i) = (\bar{y}, p_i) + t(p_0, p_i) > d_i$. On the other hand, this half-line cuts the boundary C of D at some point y_1. Then, by the assumption of the lemma, $y_1 \in \cup X_i$—a contradiction. This proves Lemma 1.

Let X be some n-dimensional half-space containing the point x_{j0}, where $j \in \{1, \ldots, m\}$ is some index. There are two possible cases: the control $u_j(t)$, $0 \leqslant t \leqslant \vartheta$ of the pursuer is subject either to an integral constraint (1.3) or to a geometric constraint (1.4). Suppose the first case occurs. We introduce the notation $Y = X \cap H(y_0, \sigma\vartheta)$.

Lemma 2. If $y(\vartheta) \in X$ and

$$Y \subset H(x_{j0}, \rho_j\sqrt{\vartheta}) \tag{2.2}$$

then a strategy of pursuer P_j exists guaranteeing the equality $x_j(\vartheta) = y(\vartheta)$.

Proof. Let $\upsilon = \upsilon(t)$, $0 \leqslant t \leqslant \vartheta$ be an arbitrary admissible control of the evader E. We define a strategy of the pursuer P_j in the time interval $[0, \vartheta]$ as follows:

$$u_j(t) = \begin{cases} (y_0 - x_{j0})/\vartheta + \upsilon(t), & 0 \leqslant t \leqslant T \\ 0, & T < t \leqslant \vartheta \end{cases} \tag{2.3}$$

where $T \in [0; \vartheta]$ is the time for which

$$\int_0^T |u_j(t)|^2\,dt = \rho_j^2$$

provided such a time exists

Let $x_{j0} = y_0$. Then it follows from (2.2) that $\sigma\vartheta \leqslant \rho_j\sqrt{\vartheta}$, and (2.3) takes the form $u_j(t) = \upsilon(t)$. Consequently

$$\int_0^\vartheta |u_j(t)|^2\,dt = \int_0^\vartheta |\upsilon(t)|^2\,dt \leqslant \vartheta\sigma^2 \leqslant \rho_j^2$$

i.e. the control $u_j(t) = \upsilon(t)$, $0 \leqslant t \leqslant \vartheta$ is admissible. It clearly guarantees that $x_j(t) = y(t)$, $0 \leqslant t \leqslant \vartheta$. Thus, the lemma is true in this case.

Now let $x_{j0} \neq y_0$. Put

$$e = (y_0 - x_{j0})/|y_0 - x_{j0}|$$

$a = \max\{(z - x_{j0}, e): z \in Y\}$, $b = \max\{(z - y_0, e): z \in Y\}$, where (x, y) is the scalar product of the vectors x and y. Note that

$$a - b = (y_0 - x_{j0}, e) = |y_0 - x_{j0}| \tag{2.4}$$

It follows from (2.2) that

$$\rho_j^2\vartheta - \sigma^2\vartheta^2 \geqslant a^2 - b^2 \tag{2.5}$$

We will show that the strategy (2.3) is admissible. Noting the inequality

$$\int_0^\vartheta (\upsilon(t), e)\,dt \leqslant b$$

which follows from the fact that $y(\vartheta) \in X$ and from (2.4) and (2.5), we deduce that

$$\int_0^\vartheta |u_j(t)|^2\,dt = \frac{1}{\vartheta}|y_0 - x_{j0}|^2 + \frac{2}{\vartheta}|y_0 - x_{j0}| \int_0^\vartheta (v(t), e)dt + \sigma^2\vartheta \leqslant$$

$$\leqslant \frac{1}{\vartheta}((a-b)^2 + 2(a-b)b + \sigma^2\vartheta^2) \leqslant \rho_j^2$$

i.e. the pursuer's strategy thus constructed is admissible.

We will now show that the strategy (2.3) guarantees the quality $x_j(\vartheta) = y(\vartheta)$. In fact

$$x_j(\vartheta) = x_{j0} + \int_0^\vartheta u_j(t)dt = y_0 - x_{j0} + x_{j0} + \int_0^\vartheta v(t)dt = y(\vartheta)$$

Lemma 2 is proved.

Now consider the case when the pursuer's control is subject to geometric constraint (1.4).

Lemma 3. If $y(\vartheta) \in X$, $\sigma \leqslant \rho$ and

$$Y \subset H(x_{j0}, \rho_j\vartheta) \tag{2.6}$$

then a strategy for pursuer P_j exists guaranteeing that $x_j(\vartheta) = y(\vartheta)$.

Proof. Let $\upsilon = \upsilon(t)$, $0 \leqslant t \leqslant \vartheta$ be an arbitrary admissible control of the evader. We define a strategy for pursuer P_j as follows:

$$u_j(t) = \begin{cases} v(t) - (v(t), e)e - er_j(t), & x_j(t) \neq y(t) \\ v(t), & \tau \leqslant t \leqslant \vartheta \end{cases} \tag{2.7}$$
$$r_j(t) = [\rho_j^2 - \sigma^2 + (v(t), e)^2]^{1/2}$$

where $\tau \in [0; \vartheta]$ is the first time at which $x_j(\tau) = y(\tau)$. Clearly, the strategy we have constructed for pursuer P_j is admissible.

If $x_{j0} = y_0$, it follows from (2.7) that $u_j(t) = \upsilon(t)$, $0 \leqslant t \leqslant \vartheta$. Then it is clear that $x_j(\vartheta) = y(\vartheta)$.

Let $x_{j0} \neq y_0$. Then by (2.4) and (2.7) we have

$$y(\tau) - x_j(\tau) = ef(\tau), \quad f(\tau) = a - b - \int_0^\tau (r_j(t) - (v(t), e))dt$$

Obviously, $f(0) = a - b = |x_{j0} - y_0| > 0$.

We now show that $f(\vartheta) \leqslant 0$. It will then be proved that $f(\tau) = 0$ for some $\tau \in [0; \vartheta]$. Consider the vector-valued function $g(t) = (\sqrt{(\rho_j^2 - \sigma^2)}; (\upsilon(t), e))$. The inequality

$$\int_0^\vartheta |g(t)|\,dt \geqslant \left|\int_0^\vartheta g(t)dt\right|$$

implies that

$$f(\vartheta) \leqslant a - b + b - [(\rho_j^2 - \sigma^2)\vartheta^2 + b^2]^{1/2} \leqslant 0$$

(we are using the inequality $a^2 - b^2 \leqslant (\rho_j^2 - \sigma^2)\vartheta^2$, which follows from (2.6)).

Consequently, for some $\tau \in [0, \vartheta]$ we have $x_j(\tau) = y(\tau)$. By the construction of the pursuer's strategy (2.7), we have $\mu_j(t) = \upsilon(t)$ for $\tau \leqslant t \leqslant \vartheta$. Hence it follows that $x_j(\vartheta) = y(\vartheta)$, Lemma 3 is proved.

We now introduce fictitious pursuers (FPs) $z_1, \dots, z_k, z_{k+1}, \dots, z_m$, whose motions are described by the equations

$$\dot{z}_i = w_i, \quad z_i(0) = x_{i0}$$

and whose controls must obey the constraints

$$\int_0^{\vartheta} |w_j(t)|^2 \, dt \leqslant \left(\rho_j + \frac{\gamma}{\sqrt{\vartheta}}\right)^2, \quad j = 1, \ldots, k$$

$$|w_j(t)| \leqslant \rho_j + \frac{\gamma}{\vartheta}, \quad j = k+1, \ldots, m$$

It can be shown that the domain of reachability of a FP z_i from the initial position x_{i0} up to time ϑ is a ball $G_i(\gamma)$. By the definition of the number γ, we have

$$H(y_0, \sigma\vartheta) \subset \cup G_i(\gamma) \tag{2.8}$$

Let

$$I = \{j: \ j \in \{1, \ldots, m\}, \ \ S(y_0, \sigma\vartheta) \cap G_j(\gamma)\} \neq \emptyset$$

It then follows from (2.8) that

$$H(y_0, \sigma\vartheta) \subset \bigcup_{j \in I} G_j(\gamma) \tag{2.9}$$

Put

$$e_i = \begin{cases} (y_0 - x_{i0}) / |y_0 - x_{i0}|, & x_{i0} \neq y_0 \\ p_0, & x_{i0} = y_0 \end{cases}$$

$$a_j = \max\{(z - x_{j0}, e_j): \ z \in S(y_0, \sigma\vartheta) \cap G_j(\gamma)\}, \ \ j \in I$$

$$X_j = \{x: \ x \in R^n, \ (x - x_{j0}, e_j) \leqslant a_j\}, \ \ j \in I$$

Clearly

$$S(y_0, \sigma\vartheta) \cap G_j(\gamma) \subset X_j, \quad j \in I$$

Hence, in view of (2.9), we have

$$y(\vartheta) \subset \bigcup_{j \in I} X_j \tag{2.10}$$

If we now recall the assumption of the theorem, $(p_0, e_j) \geqslant 0$, $j \in I$, and use Lemma 1, we obtain

$$H(y_0, \sigma\vartheta) \subset \bigcup_{j \in I} X_j$$

We define strategies for the FPs in the time interval $[0, \vartheta]$ as follows:

$$w_j(t) = \begin{cases} (y_0 - x_{j0}) / + v(t), & 0 \leqslant t \leqslant T_j \\ 0, & T_j < t, \ \ j \in I_0 \end{cases} \tag{2.11}$$

where $I_0 = I \cap \{1, \ldots, k\}$, T_j is the time for which

$$\int_0^{T_j} |w_j(t)|^2 \, dt = \left(\rho_j + \frac{\gamma}{\sqrt{\vartheta}}\right)^2$$

if such a time exists

$$w_j(t) = \begin{cases} v(t) - (v(t), \ e_j)e_j + e_j \bar{r}_j(t), & z_j(t) \neq y(t) \\ v(t), \ \ \tau_j \leqslant t \leqslant \vartheta, & j \in I_1 \end{cases} \tag{2.12}$$

$\bar{r}_j(t) = [\rho_j + \gamma/\vartheta)^2 - \sigma^2 + (\upsilon(t), e_j)^2]^{1/2}$, $I_1 = I \cap \{k+1, \ldots, m\}$, and τ_j is the first time for which $z_j(\tau_j) = y(\tau_j)$.

It follows from (2.10) that $y(\vartheta) \in X_s$ for some $s \in I$. Then, using the fact that

$$X_s \cap H(y_0, \sigma\vartheta) \subset G_s(\gamma), \quad s \in I$$

which follows from the definition of the half-space X_s, we see that the assumptions of Lemma 2 are valid if $s \in \{1, \ldots, k\} \cap I$ and those of Lemma 3 if $s \in \{k+1, \ldots, m\} \cap I$. As a result of these lemmas, the strategy for the FP, (2.11) if $s \in I_0$ or (2.12) if $s \in I_1$, guarantees the equality $z_s(\vartheta)\, y(\vartheta)$.

Thus, the strategies constructed for the FPs guarantee that $z_s(\vartheta)\, y(\vartheta)$ for some $s \in I$.

We will now prove the theorem. We construct strategies for the pursuers with the help of the strategies of the FPs

$$u_j(t) = \frac{\rho_j \vartheta^\xi}{\rho_j \vartheta^\xi + \gamma} w_j(t), \quad 0 \leqslant t \leqslant \vartheta$$

$$\xi = \begin{cases} 1/2, & j \in I_0 \\ 1, & j \in I_1 \end{cases}$$

$$u_j(t) = 0, \quad 0 \leqslant t \leqslant \vartheta, \quad j \in \{1, \ldots, m\} / I$$

It follows from the equality $z_s(\vartheta)\, y(\vartheta)$ for $s \in I_0$ that

$$|x_s(\vartheta) - y(\vartheta)| = |x_s(\vartheta) - z_s(\vartheta)| = \left| \int_0^\vartheta (u_s(t) - w_s(t) dt \right| \leqslant \frac{\gamma}{\rho_s \sqrt{\vartheta} + \gamma} \int_0^\vartheta |w_s(t)| dt \leqslant \gamma$$

(we have used the Cauchy–Bunyakovskii inequality).

If $s \in I_1$, we obtain an analogous inequality, since $|w_s(t)| \leqslant \rho_s + \gamma/\vartheta$.

Thus, the pursuers' strategies guarantee that $|x_s(\vartheta) - y(\vartheta)| \leqslant \gamma$ for some $s \in I$.

In order to complete the proof of the theorem, it remains to prove the existence of a strategy for the evader E which guarantees that

$$|x_i(\vartheta) - y(\vartheta)| \geqslant \gamma \tag{2.13}$$

for any admissible controls $u_i(t)$, $0 \leqslant t \leqslant \vartheta$.

By the definition of γ, a point $z_0 \in H(y_0, \sigma\vartheta)$ exists such that $\max|x - z_0| = \gamma$, where the maximum is taken over all

$$x \in \bigcup_{j=1}^{k} H(x_{j0}, \rho_j \sqrt{\vartheta}) \cup \bigcup_{j=k+1}^{m} H(x_{j0}, \rho_j \vartheta)$$

The evader's control is defined as follows:

$$\upsilon(t) = \sigma(z_0 - y_0)/|z_0 - y_0|, \quad 0 \leqslant t \leqslant \vartheta$$

Clearly, this control guarantees the validity of inequalities (2.13). Hence γ is indeed the value of the game. The theorem is proved.

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